

Matrices about the lattice paths

Jishe Feng¹

*Department of Mathematics, Longdong University, Qingyang, Gansu, 745000,
China*

Abstract

From the matrix point of view, we use the recursion to discuss four combinatorial numbers in terms of the integer lattice paths, this is different from Andr a's method [5]. We give four tables and matrices, and their relations, and give their matrix decompositions. By their relations, we give the explicit formulas of the special combinatorial numbers.

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1 Introduction

Many special integers can be interpreted to the numbers of path on the integer lattice in the coordinate plane, such as Catalan number and Schr oder number can be interpreted to the number of lattice paths that start at $(0, 0)$, end at (n, n) , contain no points above the line $y = x$, and are composed only of horizontal step $(0, 1)$, vertical step $(1, 0)$, and up diagonal step $(1, 1)$, i.e., \rightarrow , \uparrow , and \nearrow [1]. In paper [2], the authors discuss the ballot problem, Catalan number and their generalization in terms of p -good paths on the integer lattice, in order to get the formula, they used Andr a's reflection method [1,2] to subtract the numbers of the bad path from the total number to obtain the number of good path. Renault [3,4] described the first combinatorial proof as given by Andr a [5] and claimed that the reflection method typically misattributed to Andr a. Goulden and Serrano [6] provide a direct geometric bijection for the number of lattice paths that never go below the line $y = kx$ for a positive integer k , it uses rotation instead of reflect.

¹ Corresponding author. E-mail: gsfjs6567@126.com

From the historical point of view, the oldest are known as Pascal matrix P , and its n order minor P_n , they are as follows.

$$P = \begin{pmatrix} \binom{0}{0} & & & & \\ \binom{1}{0} & \binom{1}{1} & & & \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, P_n = \begin{pmatrix} \binom{0}{0} & & & & \\ \binom{1}{0} & \binom{1}{1} & & & \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-1} \end{pmatrix}$$

Various types of Pascal matrices are investigated in [7-10]. Shapiro [10] introduced a number triangle with its first column entries containing the Catalan numbers, Barcucci [11] set up the Catalan triangle, which is a number triangle with entries equal to $\frac{n-k+1}{n+1} \binom{n+k}{k}$.

In this paper, from the matrix point of view, we use the recursion to discuss four combinatorial numbers in terms of the integer lattice paths, this is different from Andr a's method [5]. It is the purpose of this paper to discuss four combinatorial numbers about the lattice path, and give four tables and matrices which are different from the Toeplitz matrices which possess numbers of various type arranged on the main diagonal and below [12], and give their matrix decompositions. By their relations, we give the explicit formulas of the special combinatorial numbers.

2 Rectangular lattice path and combinatorial number

For convenience to express as a matrix, we choose the vertical down line as x -axis and the horizontal line as y -axis, so the vertical step \uparrow changes to \downarrow , up diagonal step \nearrow changes to down diagonal step \searrow . For any positive integer k, r , A rectangular lattice path [1] from $(0, 0)$ to (k, r) is a path from $(0, 0)$ to (k, r) that is made up of horizontal steps \rightarrow and vertical steps \downarrow . Let $b(k, r)$ denote the number of the rectangular lattice path from $(0, 0)$ to (k, r) .

In order to get the explicit expression of the number of the rectangular lattice path, we can analyze it step by step. For each point on x -axis or y -axis, there is only one path from $(0, 0)$ to them, it is only \rightarrow or \downarrow , namely $b(k, 0) = b(0, r) = 1$. For the point $(1, 1)$, that is either from the left point $(1, 0)$ or from the above point $(0, 1)$ to it, thus the number $b(1, 1)$ is the sum of $b(1, 0)$ and $b(0, 1)$, that is $b(1, 1) = b(1, 0) + b(0, 1)$. In general for the point (k, r) , the last step of the rectangular lattice path from $(0, 0)$ to (k, r) is either from $(k, r-1)$

by \rightarrow or from $(k-1, r)$ by \downarrow to it, so there is

$$b(k, r) = b(k, r-1) + b(k-1, r). \quad (1)$$

By tabulating the number of rectangular lattice paths from $(0, 0)$ to each point, we obtain a symmetric number table (see table 1) and symmetric matrix Q .

Table 1

$k \backslash r$	0	1	2	3	4	5	6	...
0	1	1	1	1	1	1	1	...
1	1	2	3	4	5	6	7	...
2	1	3	6	10	15	21	28	...
3	1	4	10	20	35	56	84	...
4	1	5	15	35	70	126	210	...
5	1	6	21	56	126	252	462	...
6	1	7	28	84	210	462	924	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ 1 & 3 & 6 & 10 & 15 & 21 & 28 & \dots \\ 1 & 4 & 10 & 20 & 35 & 56 & 84 & \dots \\ 1 & 5 & 15 & 35 & 70 & 126 & 210 & \dots \\ 1 & 6 & 21 & 56 & 126 & 252 & 462 & \dots \\ 1 & 7 & 28 & 84 & 210 & 462 & 924 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix},$$

Theorem 1 *The number of rectangular lattice paths from $(0, 0)$ to (k, r) equals the binomial coefficient, namely $b(k, r) = \binom{k+r}{r}$.*

PROOF. *A rectangular lattice path from $(0, 0)$ to (k, r) is uniquely determined by its sequence of k horizontal steps \rightarrow and r vertical steps \downarrow , and every such sequence determines a rectangular lattice path from $(0, 0)$ to (k, r) . Hence, the number of paths equals the number of permutations of $k+r$ objects of which k are \rightarrow and s are \downarrow , that is the result [1].*

According to Pascal's formula $\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$, and the formula (1), we can get the propositions as follows.

Proposition 2 *Let $Q(p, q)$ be the element in p th row and q th column of the matrix Q , then every element of the symmetric matrix Q in first row and first column is 1, and every other element is the sum of its above element and left element.*

$$Q(p, q) = Q(p, q-1) + Q(p-1, q), \quad (2)$$

$$Q(p, q) = \sum_{i=0}^q Q(p-1, i). \quad (3)$$

Proposition 3 *Q_n can be decomposed as the product of Pascal matrix and its transposition.*

$$Q_n = P_n P_n^T \quad (4)$$

This property has been in paper [13]. By carrying out the multiplication of the matrices P_n and P_n^T , we can obtain the two identities

Proposition 4

$$\sum_{k=0}^{\min(i,j)-1} \binom{i-1}{k} \binom{j-1}{k} = \binom{i+j-2}{i-1}; \quad (5)$$

$$\sum_{k=0}^{i-1} \binom{i-1}{k}^2 = \binom{2i-2}{i-1}, \quad (6)$$

where $i = 1, 2, 3, \dots$.

3 Subdiagonal rectangular lattice path and Catalan number

We now consider rectangular lattice paths from $(0, 0)$ to (p, q) that are restricted to lie on or below the line $y = x$ in the coordinate plane whenever $p \geq q$. Brualdi [1] call such paths subdiagonal rectangular lattice paths. Let $C(p, q)$ denote the number of the subdiagonal rectangular lattice path from $(0, 0)$ to (p, q) .

We can also analyze it step by step. For each point on x -axis, there is only one path from $(0, 0)$ to them, it only \downarrow , namely $C(p, 0) = 1$. For the point $(1, 1)$, it is only from the left point $(1, 0)$ to it, thus the number $C(1, 1)$ is $C(1, 0)$, that is $C(1, 1) = C(1, 0)$. For the point $(2, 1)$, its last step is either from the left point $(2, 0)$ or from the above point $(1, 1)$ to it, namely, $C(2, 1) = C(2, 0) + C(1, 1)$, and so on, we can obtain a triangle, which is same as the triangle in [11] (see table 2).

Table 2

$p \backslash q$	0	1	2	3	4	5	6	7	...
0	1								
1	1	1							
2	1	2	2						
3	1	3	5	5					
4	1	4	9	14	14				
5	1	5	14	28	42	42			
6	1	6	20	48	90	132	132		
7	1	7	27	75	165	297	429	429	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...

Because the restriction that it must lie on or below the line $y = x$, the element in diagonal $C(n, n)$ equals the element $C(n, n-1)$. Compare the lower triangle of Table 1 and Table 2, we find that the element in Table 2 $C(p, q)$ equal the element in Table 1 $b(p, q)$ subtract the element $b(p+1, q-1)$, that is $C(p, q) = b(p, q) - b(p+1, q-1)$. This is derived from the symmetric of the Table 1, $b(p+1, q-1)$ is the number of subdiagonal rectangular paths above the line $y = x$, it is sure that it must be subtracted. By Theorem 1, $b(p, q) = \binom{p+q}{p}$, we can get

$$\begin{aligned}
C(p, q) &= b(p, q) - b(p+1, q-1) = \binom{p+q}{q} - \binom{p+1+q-1}{q-1} \\
&= \binom{p+q}{q} - \binom{p+q}{q-1} = \frac{p-q+1}{p+1} \binom{p+q}{q}.
\end{aligned}$$

Thus, we can get the following theorem.

Theorem 5 For $C(p, q)$, The number of subdiagonal rectangular lattice paths from $(0, 0)$ to (p, q) , there is [1]

$$C(p, q) = \frac{p-q+1}{p+1} \binom{p+q}{q}. \quad (7)$$

There are the same formula as (2) and (3)

$$C(p, q) = C(p-1, q) + C(p, q-1), \quad (8)$$

$$C(p+1, q) = \sum_{k=0}^q C(p, k). \quad (9)$$

It is easy found that diagonal elements and its left elements are *Catalan numbers*, this means that they are the number of subdiagonal rectangular lattice paths from $(0, 0)$ to (n, n) . From the formula (7), one obtains $C(n, n) = \frac{1}{n+1} \binom{2n}{n}$, this is the well-known Catalan number C_n . By the formula (8) and (9), we can get two identities and the recurrence formula about Catalan number as follows:

$$C_n = C(n, n-1) = \frac{1}{n} \binom{2n-2}{n-1} + \frac{3}{n+1} \binom{2n-2}{n-2}, \quad (10)$$

$$C_n = C(n, n) = \sum_{k=0}^{n-1} \frac{n-k}{n} \binom{n+k-1}{k} = \frac{1}{n+1} \binom{2n}{n}, \quad (11)$$

$$C_n = C(n, n) = \frac{2}{n+1} \binom{2n-1}{n-1} = \frac{2(2n-1)}{n+1} C_{n-1}. \quad (12)$$

4 HVD-lattice path and Delanny number

A *HVD*-lattice path [1] is a rectangular lattice path from $(0, 0)$ to (p, q) is a path from $(0, 0)$ to (p, q) that is made up of horizontal steps \rightarrow , vertical steps \downarrow and up diagonal step \searrow . Let $g(p, q)$ be the number of *HVD*-lattice paths from $(0, 0)$ to (p, q) .

We can analyze HVD-lattice path step by step. For each point on x -axis or y -axis, there is only one path from $(0, 0)$ to them, it only \rightarrow or \downarrow , namely $g(k, 0) = g(0, r) = 1$. For the point $(1, 1)$, it is either from the left point $(1, 0)$ by \rightarrow or from the above point $(0, 1)$ by \downarrow , or from the point $(0, 0)$ by \searrow to it, thus the number $g(1, 1)$ is the sum of $g(1, 0)$, $g(0, 1)$, and $g(0, 0)$, that is $g(1, 1) = g(1, 0) + g(0, 1) + g(0, 0)$. In general for the point (k, r) , the last step of the *HVD*-lattice path is either from $(k, r-1)$ by \rightarrow or from $(k-1, r)$ by \downarrow , or from the $(k-1, r-1)$ by \searrow to it, so there is

$$g(k, r) = g(k, r-1) + g(k-1, r) + g(k-1, r-1). \quad (13)$$

this is as same as in paper [14]. In [15], $g(k, r)$ is called as *Delanny number*. By tabulating the number of *HVD*-lattice paths from $(0, 0)$ to each point, we obtain a symmetric number table (see table 3) which is as same as the Table 1 in paper [14] and symmetric matrix K as follows.

Table 3

$p \backslash q$	0	1	2	3	4	5	6	...
0	1	1	1	1	1	1	1	...
1	1	3	5	7	9	11	13	...
2	1	5	13	25	41	61	85	...
3	1	7	25	63	129	231	377	...
4	1	9	41	129	321	681	1289	...
5	1	11	61	231	681	1683	3653	...
6	1	13	85	377	1289	3653	8989	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & \dots \\ 1 & 5 & 13 & 25 & 41 & 61 & 85 & \dots \\ 1 & 7 & 25 & 63 & 129 & 231 & 377 & \dots \\ 1 & 9 & 41 & 129 & 321 & 681 & 1289 & \dots \\ 1 & 11 & 61 & 231 & 681 & 1683 & 3653 & \dots \\ 1 & 13 & 85 & 377 & 1289 & 3653 & 8989 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Theorem 6 [1] Let $r \leq \min\{p, q\}$. Then

$$g(p, q) = \sum_{r=0}^{\min\{p, q\}} \binom{p+q-r}{p-r \quad q-r \quad r} = \sum_{r=0}^{\min\{p, q\}} \frac{(p+q-r)!}{(p-r)!(q-r)!r!} \quad (14)$$

where $\binom{p+q-r}{p-r \quad q-r \quad r}$ is the multinomial number [1].

Theorem 7 [14]

$$g(p, q) = \sum_{\alpha=0}^p \binom{p}{\alpha} \binom{q+\alpha}{p} \quad (15)$$

Theorem 8 [14]

$$g(p, q) = \sum_{\alpha=0}^{\min\{p,q\}} \binom{p}{\alpha} \binom{q}{\alpha} 2^\alpha \quad (16)$$

We can rewrite (16) to matrix format, there is the following proposition.

Proposition 9 *The first n rows and n columns K_n is the product of Pascal matrix P_n , diagonal matrix D_n and its transposition of Pascal matrix P_n^T .*

$$K_n = P_n D_n P_n^T \quad (17)$$

where $D_n = \text{diag}(1, 2, 2^2, 2^3, \dots, 2^{n-1})$.

5 Subdiagonal HVD-lattice path and Schröder number

We now consider subdiagonal *HVD*-lattice path from $(0, 0)$ to (p, q) that is restricted to lie on or below the line $y = x$ in the coordinate plane whenever $p \geq q$. Let $R(p, q)$ be the number of the subdiagonal *HVD*-lattice path from $(0, 0)$ to (p, q) .

When we analyze subdiagonal *HVD*-lattice path step by step, we obtain a triangle (see table 4)

Table 4

$p \backslash q$	0	1	2	3	4	5	6	7	...
0	1								
1	1	2							
2	1	4	6						
3	1	6	16	22					
4	1	8	30	68	90				
5	1	10	48	146	304	394			
6	1	12	70	264	714	1412	1806		
7	1	14	96	430	1408	3534	6752	8558	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

Compare the lower triangle of Table 3 and Table 4, we find that the element in Table 4 $R(p, q)$ equal the element in Table 3 $g(p, q)$ subtract the element $g(p+1, q-1)$, that is $R(p, q) = g(p, q) - g(p+1, q-1)$. This is derived from

the symmetric of the Table 3, $g(p+1, q-1)$ is the number of subdiagonal HVD-lattice paths above the line $y = x$, it is sure that it must be subtracted. By Theorem 6, 7, 8, we can get

Theorem 10 *Let p and q be positive integers with $q \geq p$, and let r be a nonnegative integer with $r \leq q$. Then*

$$\begin{aligned} R(p, q) &= g(p, q) - g(p+1, q-1) \\ &= \sum_{r=0}^{\min\{p, q\}} \binom{p+q-r}{p-r \quad q-r \quad r} - \sum_{r=0}^{\min\{p+1, q-1\}} \binom{p+q-r}{p+1-r \quad q-1-r \quad r}, \end{aligned} \quad (18)$$

$$\begin{aligned} R(p, q) &= g(p, q) - g(p+1, q-1) \\ &= \sum_{r=0}^p \binom{p}{r} \binom{q+r}{p} - \sum_{r=0}^{p+1} \binom{p+1}{r} \binom{q-1+r}{p+1}, \end{aligned} \quad (19)$$

$$\begin{aligned} R(p, q) &= g(p, q) - g(p+1, q-1) \\ &= \sum_{r=0}^{\min\{p, q\}} \binom{p}{r} \binom{q}{r} 2^r - \sum_{r=0}^{\min\{p+1, q-1\}} \binom{p+1}{r} \binom{q-1}{r} 2^r, \end{aligned} \quad (20)$$

$$R(p, q) = \sum_{r=0}^q \frac{p-q+1}{p-r+1} \binom{p+q-r}{p-r \quad q-r \quad r}. \quad (21)$$

where (21) is in [1].

We now suppose that $p = q = n$. The subdiagonal HVD-lattice paths from $(0, 0)$ to (n, n) are called Schröder paths [1, 16, 17]. The *large Schröder number* S_n is the number of Schröder paths from $(0, 0)$ to (n, n) . Thus, we have

$$S_n = R(n, n) = \sum_{r=0}^n \frac{1}{n-r+1} \binom{2n-r}{n-r \quad n-r \quad r}, \quad (22)$$

$$S_n = R(n, n) = 2^n + \sum_{r=0}^{n-1} \left[\frac{r 2^r}{n-r+1} \binom{n}{r} \binom{n-1}{r} \right]. \quad (23)$$

By the formula (18-21), we obtain following proposition.

Proposition 11 *The numbers $R(p, q)$ are given by the recurrence relation*

$$R(p, q) = R(p-1, q-1) + R(p-1, q) + R(p, q-1), \quad (24)$$

$$S_n = R(n, n) = R(n-1, n-1) + R(n, n-1), \quad (25)$$

$$R(p, q) = 2[R(p-1, 0) + R(p-1, 1) + \cdots + R(p-1, q-1)] + R(p-1, q), \text{ if } p \neq q, \quad (26)$$

$$R(n, n) = 2[R(n-1, 0) + R(n-1, 1) + \cdots + R(n-1, n-1)], \quad (27)$$

$$S_n = S_{n-1} + S_0 S_{n-1} + S_1 S_{n-2} + S_2 S_{n-3} + \cdots + S_{n-1} S_0 \quad (28)$$

$$(n+2)S_{n+2} = 3(2n+1)S_{n+1} - (n-1)S_n. \quad (29)$$

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